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Design Strategy with a Class of Structural Reliability Models of High Dimension

Hector A. Jensen*
Santa Maria University, Casilla 110-V, Valparaiso, Chile

The reliability-based optimization of structural systems with reliability models of high dimension is considered. The design process is written as a two-level nonlinear constrained optimization problem. The top-level includes the overall optimization in the design variables, whereas the sublevel problem corresponds to the failure probability estimates. Attention is directed toward problems in which the structural system is modeled as a series system of a large number of secondary failure elements. The problem of calculating the system failure probability is transformed to calculating the failure probability of the secondary failure events. It is assumed that the secondary failure events are relatively simple to describe and characterized by their design points. The probability content of the individual secondary failure events is approximated by a first-order estimate. The interaction of the secondary failure events is considered by an efficient importance sampling technique that is integrated into the optimization process. Several issues regarding the numerical implementation of the methodology are addressed. The feasibility of the proposed method is demonstrated by the optimization of a steel column loaded by a set of axial loads with stability and material failure modes.

Nomenclature

$C(\cdot)$	=	total cost function
E	=	modulus of elasticity
F	=	failure event
F_j	=	secondary failure event
f_y	=	yield stress
$f(\cdot)$	=	importance sampling density
$g_j(\cdot)$	=	limit state function
$\tilde{g}_j(\cdot)$	=	limit state function in the normalized space
$I_F(\cdot)$	=	indicator function
k_d	=	knockdown factor
k_I	=	imperfection parameter
$P_F(\cdot)$	=	probability of failure of the system
$\tilde{P}_F(\cdot)$	=	estimator of the failure probability
P_F^*	=	specified level of failure
$p(\cdot)$	=	probability density function
$Q(\cdot)$	=	importance sampling quotient
$q(\cdot)$	=	one-dimensional normal probability density function
u_1	=	uniform random variable
$\{z\}$	=	vector of design variables
β_j	=	reliability index
ν	=	Poisson ratio
ρ	=	coefficient of correlation
$\Phi(\cdot)$	=	standard normal distribution function
$\phi(\cdot)$	=	standard multidimensional normal probability density function
Ω	=	parameter space
$\{\omega\}$	=	vector of uncertain parameters
$\{\varpi\}$	=	vector of uncorrelated normal variables
$\{\varpi\}_j^*$	=	design point

Introduction

THE field of optimization has recently seen an advancement from deterministic strategies to nondeterministic formulations to model and assess the effect of uncertainty.[1–6] In general, very few real engineering problems are void of uncertainty, such as variation in material characteristics, loading conditions, geometric properties, manufacturing precision, etc. Traditionally, many uncertainties are removed or handled through crude safety factor methods, which do not offer insight into the effects of the uncertainty and the actual margin of safety of the design. Recently, probabilistic formulations that couple powerful optimization techniques with probabilistic analysis and design methods have been developed.[2,7–9] These problems are called reliability-based optimization problems and may be applied to realistic systems of practical interest such as complex aerospace, civil, and mechanical systems. In addition, reliability-based optimization can produce designs that are more robust than the deterministic optimal designs because the uncertainties are considered explicitly during the optimization process. In deterministic optimization, the design obtained is optimal for the specific set of system parameter values used during the optimization. In this manner, the results are designs that achieve higher performance, from a reliability viewpoint, than those achieved using traditional deterministic optimization. In general, the numerical implementation of reliability-based optimization is much more intensive than deterministic optimization because reliability estimation requires the time consuming calculation of multidimensional probability integrals.

The reliability-based optimization problem can be characterized as a two-level optimization problem. The top level is the overall optimization in the decision or design variables. The sublevel is the reliability estimation level in which a multidimensional integral over the uncertain parameter space has to be evaluated. Before a reliability-based optimization problem can be solved, methods to estimate reliability must be accessible. Two classes of methods are widely used to compute structural reliability. The first class consists of first- and second-order reliability methods.[10,11] These methods have been developed to provide economical computational tools for approximating structural reliability. Traditionally, first-order reliability techniques have been used in the context of reliability-based optimization problems. This has been done from computational as well as optimization points of view. However, in certain situations such as highly concave limit state surfaces, convex limit state surfaces, or other particular cases, linear approximation techniques may be prone to considerable errors.

The second class of methods consists of Monte Carlo sampling techniques.[12] The basic Monte Carlo method is a numerical sampling technique that simulates a process involving realization of random variables and determination if failure occurs for each simulation. The method is robust to the type and dimension of the problem,

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*Associate Professor, Department of Civil Engineering. Member AIAA.

but it is not suitable for finding small probabilities. Finding small probabilities requires information from rare samples corresponding to failures, and on average, it would require many samples before a failure occurs. To reduce the number of simulations, a variance reduction technique called importance sampling has been introduced in recent years.[13,14] The idea is to carry out Monte Carlo simulation with samples having a higher rate of falling in the failure domain because only these samples contribute to the evaluation of the failure probability. In this class of methods, an importance sampling density function is used to reduce the variance in the estimate during the simulation procedure. The efficiency of the method relies in a proper choice of the importance sampling density, which requires some knowledge about the failure domain. Schemes for constructing importance sampling density functions, such as those based on design points or presamples, have been found to be useful.[15–18]

Reliability-based optimization can in principle be applied to a wide range of problems. However, the reliability-based design of complex structural systems may involve high-dimensional uncertain parameter spaces and complex failure regions. The numerical implementation of this type of problems remains a challenging problem due to its complexity. On the other hand, the ability to perform reliability-based design of complex real systems is quite compelling. The objective of the present paper is to develop a design strategy with a class of structural reliability models of high dimension. First, the formulation of the problem is presented. Next, the characterization of the system reliability is considered. Several issues regarding the numerical implementation of the formulation are discussed. Finally, an example problem is presented to demonstrate the efficiency of the proposed methodology.

Formulation

The design process is an iterative procedure where a preliminary design is cycled through different stages to achieve a design that is optimum in some chosen sense. Let the vector $\{z\}$, z_i, $i = 1, \ldots, n$, represents the vector of structural design variables. The design variables are the parameters of the design that are selected to be varied during the optimization process. The uncertain parameters modeling the state of nature of the system are collected in a vector $\{\omega\} \in \Omega$, ω_i, $i = 1, \ldots, m$, with Ω a subset of R^m. Its components can model uncertainty related to physical model parameters such as loads, properties, and behavior. The uncertain parameters are modeled using a prescribed joint probability density function $p(\{\omega\})$. The decision maker has to choose the strategy $\{z\}$ maximizing or minimizing a given objective. For example, the problem can be formulated as an optimization problem, where the total expected cost is minimized while satisfying constraints that ensure safe and reliable operation. Then, the optimal design process can be written as a two-level nonlinear constrained optimization problem of the following form:

Find the set of design variables $\{z\}$ that minimizes

$$C(\{z\}) \tag{1a}$$

subject to

$$P_F(\{z\}) \leq P_F^*, \qquad \{z\} \in \Sigma \tag{1b}$$

where $C(\{\cdot\})$ is the objective function and Σ is the set that contains the side constraints for the vector of design variables $\{z\}$. The top-level optimization problem is the overall optimization in the design variables $\{z\}$, whereas the sublevel problem corresponds to the failure probability estimates. The two types of variables, that is, the vector of design variables $\{z\}$ and the vector of uncertain system parameters $\{\omega\}$, are nested in the problem. When reliability calculations are performed, it is for fixed values of the design variables, and when the new values of the design variables are calculated at the top level, it is for a given level of reliability. This interaction between the optimizer and the probability calculations is repeated until the process of moving from one design to an improved design converges. Note that the design variables can also be considered as random variables to allow, for example, construction variability. In the present formulation, however, the design variables are treated as deterministic parameters to simplify the presentation of the proposed design strategy. The methodology can be extended in a direct manner for the case of design problems having design parameters related to random variables.

In this formulation, attention is directed toward problems in which the structural system is modeled as a series system of a large number of secondary failure elements. Then, the failure event of interest F is defined as

$$F = \bigcup_{j=1}^{l} F_j$$

where l is the number of failure events and F_j, $j = 1, \ldots, l$, are the secondary failure events. The failure event F can be considered as the failure of a system with l subsystems connected in series, and the system failure probability at the design $\{z\}$ is written as

$$P_F(\{z\}) = P(\{\omega\} \in F) = \int_{\Omega} I_F(\{\omega\}) p(\{\omega\})\, \mathrm{d}\{\omega\} \tag{2}$$

where $I_F(\{\cdot\})$ is the indicator function, that is, $I_F(\{\omega\}) = 1$ if $\{\omega\} \in F$ and $I_F(\{\omega\}) = 0$ otherwise, and all other terms have already been defined. Note that many failure events encountered in engineering applications can be defined as the union of some component failure events. General system reliability models consisting in series systems of parallel systems can also be considered in the formulation. In that case, each subsystem of the series system consists of several components connected in parallel. Finally, additional constraints related to general design requirements can also be included in the optimization problem (1).

Secondary Failure Events

The original problem of calculating the probability of failure of the system $P_F(\{z\})$ at the design $\{z\}$, which may be prohibitively expensive, is first reduced to calculating the probability of failure of the secondary failure events $P_{F_j}(\{z\})$, $j = 1, \ldots, l$. Each secondary failure event F_j is assumed to be defined by a single-failure mode, and it is described by a limiting state function $g_j(\{\omega\} \mid \{z\})$ that divides the space into failure states $g_j(\{\omega\} \mid \{z\}) \leq 0$ and safe states $g_j(\{\omega\} \mid \{z\}) > 0$. The corresponding probability of failure of the secondary failure event, F_j, at the design $\{z\}$ is written as an m-fold integral over the parameter space Ω:

$$P_{F_j}(\{z\}) = P(\{\omega\} \in F_j) = \int_{g_j(\{\omega\} \mid \{z\}) \leq 0} p(\{\omega\})\, \mathrm{d}\{\omega\} \tag{3}$$

where $F_j = \{\{\omega\} \in \Omega : g_j(\{\omega\} \mid \{z\}) \leq 0\}$ is a secondary failure event and $g_j(\cdot)$ is the corresponding limit state function. Equivalently, the m-fold integral can be written in the standardized normal space if the basic variables ω_i, $i = 1, \ldots, m$, are not normally distributed. In this case, a one-to-one transformation $T: \{\omega\} \rightarrow \{\varpi\}$, where the variables ϖ_i, $i = 1, \ldots, m$ are uncorrelated and standardized normally distributed, is introduced.[11,19] Practical experience indicates that the numerical implementation of such a transformation is feasible in many situations. Then, the limit state function $g_j(\cdot)$ in $\{\omega\}$ space is mapped on the corresponding limit state function $\bar{g}_j(\cdot)$ in $\{\varpi\}$ space, and the probability of failure of the secondary failure event, F_j, at the design $\{z\}$ can be written as

$$P_{F_j}(\{z\}) = \int_{\bar{g}_j(\{\varpi\} \mid \{z\}) \leq 0} \phi(\{\varpi\})\, \mathrm{d}\{\varpi\} \tag{4}$$

where $\phi(\{\varpi\})$ is the standard m-dimensional normal probability density function.

In general, the evaluation of the probability integral (4) cannot be efficiently evaluated by means of direct numerical integration if the dimension m is not small. Also, the analytical evaluation of the multidimensional integral can be found for only a very limited number of simple systems. This leads to the need for approximate approaches. To evaluate the probability of failure of the secondary

failure events, it is first assumed that the secondary failure regions F_j, $j = 1, \ldots, l$, are relatively simple to describe. In this context, a failure region is considered to be simple to describe if its limit state surface $\bar{g}_j(\{\varpi\} \mid \{z\}) = 0$ is well approximated by the tangent hyperplane around the point on the failure surface closest to the origin. Note that this is the case encountered in many problems of practical relevance. In these cases, first-order reliability estimates provide a good approximation to $P_{F_j}(\{z\})$ (Ref. 11). The implementation to follow is limited to situations in which the preceding condition is satisfied. The cases in which linear approximation techniques produce inaccurate estimates for the probability of failure of the secondary events will be discussed later.

In the first-order reliability method, the failure probability $P_{F_j}(\{z\})$ at the design $\{z\}$ is approximated by $P_{F_j}(\{z\}) = \Phi(-\beta_j)$, where β_j is the reliability index that corresponds to the smallest distance from the origin to a point on the failure surface. Such a point, $\{\varpi\}_j^*$, is called the design point, and it is the solution of the minimization problem

$$\text{minimize}_{\{\varpi\}} \|\{\varpi\}\|^2 = \{\varpi\}^T\{\varpi\} \tag{5}$$

subject to $\bar{g}_j(\{\varpi\} \mid \{z\}) = 0$. The approximation of the failure probability given in terms of the reliability index β_j corresponds to the linearization of the failure surface, $\bar{g}_j(\cdot \mid \{z\}) = 0$, at the design point $\{\varpi\}_j^*$.

Interaction of Secondary Failure Events

Under the conditions specified in the preceding section, it is clear that the complexity of the current problem lies in the interaction of the secondary failure events F_j, $j = 1, \ldots, l$, in forming the system failure event

$$F = \bigcup_{j=1}^{l} F_j$$

because the secondary failure events are described in a relatively simple way. To consider the interaction of the secondary failure events, a generalized system reliability index[11] can be introduced to approximate the failure probability $P_F(\{z\})$. In this case, the generalized reliability index is defined in terms of the l-dimensional standardized normal distribution function. Except for special cases, the multidimensional standard normal distribution function must be estimated by an approximation.[20–22] One of the approaches is to approximate the failure region by the polyhedral set bounded by the tangent hyperplanes at the design points. The corresponding failure probability can be determined from various formulas for probability contents in polyhedral sets. In general, an important disadvantage of these approximate methods is that no estimation of the error made is available, and so it is difficult to validate the results. This difficulty is highly aggravated for the case of high-dimensional problems, which is the class of systems considered here. Then, the use of standard system reliability methods for estimating reliability of series systems of high dimension is not appropriate in the context of this formulation. An approach that is robust to the dimension of the problem and computationally efficient is considered here.

Reliability Estimation

An efficient importance sampling technique is used in this work to evaluate the system failure probability $P_F(\{z\})$ at the design $\{z\}$. The basic idea of importance sampling methods is to shift the sampling density toward the failure region F to produce more samples lying in F. In the importance sampling procedure, the system failure probability $P_F(\{z\})$ can be rewritten in the standard normal space as

$$P_F(\{z\}) = \int_{\bar{\Omega}} \frac{\phi(\{\varpi\}) I_F(\{\varpi\})}{f(\{\varpi\})} f(\{\varpi\})\, \mathrm{d}\{\varpi\} \tag{6}$$

where $\phi(\{\varpi\})$ is, as before, the standard m-dimensional normal probability density function, $f(\{\varpi\})$ is the importance sampling density chosen to reduce the statistical error of the estimate, and $\bar{\Omega}$ is the standard m-dimensional normal space. The efficiency of these methods relies on a proper choice of the importance sampling density, which requires some knowledge about the failure region. Because the secondary failure events F_j, $j = 1, \ldots, l$, are characterized by their design points $\{\varpi\}_j^*$, $j = 1, \ldots, l$, it seems natural to construct the importance sampling density $f(\cdot)$ based on the design points to account for the contributions from the secondary failure regions. The conventional choice for an importance sampling density using design points consists in a weighted sum of normal probability density functions among the design points.[14] However, this choice may not be efficient in terms of its numerical implementation when the number of design points is large. The importance sampling density to be used in this formulation corresponds to the methodology proposed in Ref. 23. In Ref. 23, an efficient importance sampling was developed for computing the first excursion failure probability problem of linear dynamic systems subjected to Gaussian white noise excitation.

Importance Sampling Technique

The methodology developed in Ref. 23 is adapted and integrated into the proposed optimal design process. For the sake of completeness and continuity, some of the basic ideas of that methodology are presented in this section. The importance sampling density is constructed based on the conditional probability density functions of the secondary failure events F_j, $j = 1, \ldots, l$. The conditional distribution of the input random vector transformed to the standard normal space, $\phi(\{\varpi\}|F_j)$, given that it lies in the secondary failure region F_j, is just the original normal probability density function confined to F_j and normalized by the probability content of F_j, that is, $\phi(\{\varpi\}|F_j) = \phi(\{\varpi\}) I_{F_j}(\{\varpi\}) / P_{F_j}(\{z\})$, where $P_{F_j}(\{z\})$ is the probability content of F_j at the design $\{z\}$ and all other terms have already been defined. The importance sampling density $f(\{\varpi\})$ is defined as a weighted sum of the conditional probability density functions $\phi(\{\varpi\}|F_j)$, $j = 1, \ldots, l$; thus,

$$f(\{\varpi\}) = \sum_{j=1}^{l} w_j \phi(\{\varpi\}|F_j) = \sum_{j=1}^{l} w_j \frac{\phi(\{\varpi\}) I_{F_j}(\{\varpi\})}{P_{F_j}(\{z\})} \tag{7}$$

where $w_j \geq 0$ is the weight associated with the secondary event F_j, and

$$\sum_{j=1}^{l} w_j = 1$$

The weight w_j controls the relative frequency of samples simulated from $\phi(\{\varpi\}|F_j)$ and may be chosen to reflect the relative importance of the secondary failure event F_j in contributing to the failure probability $P_F(\{z\})$. The weights are defined to be proportional to $P_{F_j}(\{z\})$ and given by

$$w_j = P_{F_j}(\{z\}) \Big/ \sum_{i=1}^{l} \Phi(-\beta_i), \qquad j = 1, \ldots, l$$

When this value is substituted into Eq. (7) and $f(\{\varpi\})$ is used as the importance sampling density, the probability of failure $P_F(\{z\})$ can be estimated by simulation as

$$P_F(\{z\}) \approx \tilde{P}_F(\{z\}) = \sum_{j=1}^{l} \Phi(-\beta_j) \times \frac{1}{N} \sum_{r=1}^{N} \left[1 \Big/ \sum_{j=1}^{l} I_{F_j}(\{\varpi\}_r) \right] \tag{8}$$

where $\{\varpi\}_r$, $r = 1, \ldots, N$ are independent identically distributed samples simulated according to $f(\cdot)$. Note that the whole failure region F is covered by the support of the importance sampling density, and therefore, the contribution from all parts of the failure region is accounted for in the estimator. In this manner, the estimator $\tilde{P}_F(\{z\})$ is an unbiased estimator of the failure probability $P_F(\{z\})$.

Implementation

Simulation Procedure

The estimation of the probability of failure $P_F(\{z\})$ requires N samples to be simulated according to the importance sampling density $f(\cdot)$. This function is defined in terms of a weighted sum of the conditional probability density functions, $\phi(\{\varpi\}|F_j)$, of the secondary failure events F_j, $j=1,\ldots,l$. Therefore, samples distributed as $\phi(\{\varpi\}|F_j)$ need to be simulated during the procedure. To simulate a sample $\{\varpi\}_r$, $r=1,\ldots,N$, a random index J from the set $j=1,\ldots,l$ with corresponding probabilities w_j, $j=1,\ldots,l$, is first drawn. Because the limit state surface $\bar{g}_J(\{\varpi\}\,|\,\{z\})=0$ of the secondary failure region F_J is well approximated by its tangent hyperplane at the design point, the representation of the conditional vector $\{\varpi\}_r$ distributed as $\phi(\{\varpi\}|F_J)$ can be approximated as

$$\{\varpi\}_r = \alpha(\{\varpi\}_J^*/\|\{\varpi\}_J^*\|) + \{\psi\}_J \tag{9}$$

where $\{\varpi\}_J^*$ is the corresponding design point of the secondary failure region F_J, α is a standard normal random variable conditional on $\alpha \geq \beta_J$, and $\{\psi\}_J$ is a standard normal vector orthogonal to $\{\varpi\}_J^*/\|\{\varpi\}_J^*\|$. The vector $\{\psi\}_J$ can be expressed as a linear combination of the vector $\{\varpi\}_J^*/\|\{\varpi\}_J^*\|$ and a simulated m-dimensional standard normal vector with independent components $\{\varpi\}$ as $\{\psi\}_J = \{\varpi\} - \langle\{\varpi\}, \{\varpi\}_J^*/\|\{\varpi\}_J^*\|\rangle\{\varpi\}_J^*/\|\{\varpi\}_J^*\|$, where $\langle\cdot,\cdot\rangle$ is the inner product operation. The probability density function $q(\cdot)$ of α is just the normal probability density function restricted to the region $G_J=\{\alpha : \alpha \geq \beta_J\}$ and normalized by the probability content of G_J, that is, $q(\alpha|G_J) = q(\alpha)I_{G_J}(\alpha)/\Phi(-\beta_J)$, where $q(\cdot)$ is the one-dimensional normal probability density function. It is easy to show that α can be determined as $\alpha = \Phi^{-1}[u_1 + (1-u_1)\Phi(\beta_J)]$, where u_1 is a simulated uniform random variable on [0,1] (Ref. 23). When the representation of $\{\psi\}_J$ is substituted into Eq. (9), $\{\varpi\}_r$ can be represented in the form

$$\{\varpi\}_r = \{\varpi\} + \left[\alpha - \left\langle\{\varpi\}, \frac{\{\varpi\}_J^*}{\|\{\varpi\}_J^*\|}\right\rangle\right]\frac{\{\varpi\}_J^*}{\|\{\varpi\}_J^*\|} \tag{10}$$

This representation of $\{\varpi\}_r$ allows an efficient simulation of samples distributed as the conditional probability density function of the secondary failure event F_J.

Evaluation of the Failure Probability Estimate

The estimator $\tilde{P}_F(\{z\})$ in Eq. (8) is the product of

$$\sum_{j=1}^{l}\Phi(-\beta_j)$$

and the average over N samples of the importance sampling quotient,

$$Q(\{\varpi\}) = 1\Big/\sum_{j=1}^{l} I_{F_j}(\{\varpi\})$$

For each sample $\{\varpi\}_r$, the evaluation of $Q(\{\varpi\}_r)$ involves the evaluation of each $I_{F_j}(\{\varpi\}_r)$, $j=1,\ldots,l$. This can be done efficiently by computing the projection of the sample vector $\{\varpi\}_r$ over the unit vector in the direction of the design point $\{\varpi\}_j^*$ with reliability index β_j, that is,

$$I_{F_j}(\{\varpi\}_r) = \begin{cases} 1, & \text{if}\langle\{\varpi\}_r, \left(\{\varpi\}_j^*\Big/\left\|\{\varpi\}_j^*\right\|\right)\rangle \geq \beta_j \\ 0, & \text{otherwise} \end{cases} \tag{11}$$

Thus, no additional evaluation of the limit state functions $\tilde{g}_j(\cdot|\{z\})$, $j=1,\ldots,l$, is required during the simulation procedure. The importance sampling density given in Eq. (7) is constructed with the contribution of all the conditional probability density functions $\phi(\{\varpi\}|F_j)$, $j=1,\ldots,l$. It is clear that using more failure events increases the computational effort of the formulation, and it is often sufficient to use only those that are important. The importance of the failure events can be measured by their probability content, that is, $\Phi(-\beta_j)$. This measure can be used to determine the number of conditional probability density functions to be included in the construction of the important sampling density.

Sublevel and Top-Level Optimization Problems

The solution of the sublevel and top-level optimization problems can be obtained by any general nonlinear optimization algorithm that is able to treat equality and inequality constraints. The algorithm used in this work is a sequential quadratic programming technique, where the gradients of the objective function and constraint are computed by finite differences.[2,24] The optimization scheme has been shown to be efficient in terms of its convergence properties, accuracy, and robustness for the example problem presented in the paper. Additional validation calculations performed for a series of numerical experiments showed a similar behavior. The convergence criteria used in the sublevel and top-level optimization processes is based on the relative change of the objective function between two consecutive iterations. The procedures converge if such change is less than to a given tolerance (10^{-3} in this implementation).

Additional Considerations

In the preceding formulation, it was assumed that the first-order reliability estimation provides a good approximation to the probability of failure of the secondary failure events F_j, $j=1,\ldots,l$. For example, if the radius of curvature of the secondary failure surface at the design point is not large compared to the reliability index or when there exist several local design points with similar contribution to the reliability integrals, the first-order reliability estimate can deviate significantly from the exact value. In those cases, the probability of failure of the secondary failure events must be evaluated by alternative methods to increase the range of applicability of the formulation. For example, better approximations can be obtained by improved approximations of the limit state surfaces such as quadratic or polyhedral surfaces.[11] If these approximations prove to be inefficient, techniques such as importance sampling procedures based on design points, second-order asymptotic approximations, or adaptive presamples may offer a feasible and accurate estimate for the probability of failure of the secondary failure events.[14,15,17,25] In the schemes based on design points or maximum likelihood points, the first stage consists in solving an optimization problem. If the problem has several optima, a global optimization algorithm[26,27] should be used to find all of the important local optima. Other probabilistic methods such as response surface methodologies, controlled Monte Carlo, and subset simulation may also be used in estimating the probability of failure of the secondary failure events.[28–30] Even though these alternative methods can in principle be used during the optimization process, further numerical experience is needed to assess the feasibility of the proposed implementation under these general conditions.

Application

A relatively simple structural system with a damage model of high dimension is chosen as a test problem to emphasize some different numerical aspects of the failure probability estimation in the context of a reliability-based optimization problem. In particular, the efficiency of the methodology to calculate the interaction of the secondary failure events during the design process is investigated. The problem consists of a tubular steel column loaded by a set of axial loads p_i, $i=1,\ldots,100$, distributed in time, with a length $h=25.0$ m, an initial diameter $d=1.5$ m, an initial thickness $e=11.00\times10^{-3}$ m, and that is simply supported, as shown in Fig. 1. The column is optimized with d and e as optimization variables, that is, $\{z\}^t=\langle d, e\rangle$. The objective of the design is to minimize the initial cost given by $C(d,t)=C_I h\pi de$, where C_I is an initial cost taken as $2.0\times10^4/\text{m}^3$. The requirements for the system reliability are related to limit state functions for material yield, local buckling, and global buckling. The material yield limit state

Table 1 Statistic characteristics

Basic variable	Expected value	COV
Modulus of elasticity E[a]	2.1×10^{11} [N/m^2]	0.05
Yield stress f_y[a]	6.5×10^{8} [N/m^2]	0.05
Knockdown factor k_d[b]	0.54	0.16
Imperfection parameter k_I[b]	0.49	0.10

[a]Lognormal. [b] Normal.

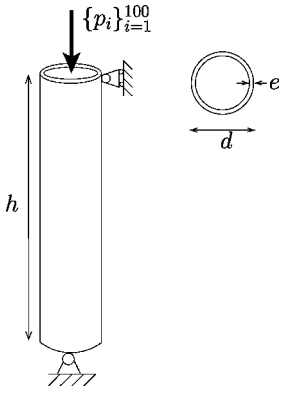

Fig. 1 Tubular steel column loaded by a set of axial loads.

function for the load p_i is given by

$$g_{1i} = f_y - (p_i/\pi de) \tag{12}$$

where f_y is the yield stress. On the other hand, the local buckling limit state function for the load p_i is written as[31]

$$g_{2i} = \tfrac{1}{2}[1 + \exp(-\lambda_b)]f_y - (p_i/\pi de) \tag{13}$$

where $\lambda_b = \sqrt{[f_y/(\theta_b S_{\text{el}})]}$ is the relative slenderness ratio, with $S_{\text{el}} = 2Ee/\{d\sqrt{[3(1-\nu^2)]}\}$ being the critical stress and $\theta_b = k_d/[\sqrt{(1+0.005d/e)}]$ a correction factor taking imperfections into account with the knockdown factor k_d. Finally, the global buckling limit state function for the load p_i is defined as

$$g_{3i} = \left(\gamma - \sqrt{\gamma^2 - 1/\lambda_e^2}\right) f_y - (p_i/\pi de) \tag{14}$$

where γ and λ_e are auxiliary parameters that depend on the specific design and production method. For a cold-formed, welded, and nonstress relieved steel cylinder, the parameters take the form $\gamma = 1/(2\lambda_e^2)[\lambda_e^2 + k_I(\lambda_e - 0.2) + 0.8]$ and $\lambda_e = h/(0.35d\pi)\sqrt{(f_y/E)}$, where k_I is an imperfection parameter.[31]

The uncertainties of the tubular steel column system are connected with the yield stress f_y, the modulus of elasticity E, the knockdown factor k_d for the local buckling limit state, and the imperfection parameter k_I for the global buckling limit state. These parameters are modeled as independent random variables. The parameters k_d and k_I are assumed normally distributed, whereas the parameters E and f_y are lognormally distributed. The mean values and coefficient of variations (COV) of these basic random variables are given in Table 1. On the other hand, the set of loads p_i, $i = 1, \ldots, 100$, is modeled as a normal random vector with second-order representation given by

$$E_t(p_i) = \mu_p$$

$$C_{ij} = E_t\big(\big[p_i - E_t(p_i)\big]\big[p_j - E_t(p_j)\big]\big) = \begin{cases} \sigma_p^2, & \text{if } i = j \\ \rho\sigma_p^2, & \text{if } i \neq j \end{cases} \tag{15}$$

where $E_t(\cdot)$ is the operator of mathematical expectation, C_{ij} is the ij component of the corresponding covariance matrix, μ_p is the mean value, σ_p^2 is the variance, and ρ is the coefficient of correlation of the components of the random vector. A mean value equal to $\mu_p = 10^7$ N and a COV equal to $\sigma_p/\mu_p = 0.2$ is considered in this case. To study the effect of correlation in the axial loads on the final design, three values of the coefficient of correlation are considered: $\rho = 1.0$, 0.8, and 0.0. In the first case, the elements of the axial load vector are fully correlated, whereas they are completely uncorrelated in the third case. In the second case, the elements of the random vector can be considered as strongly correlated. By means of the spectral decomposition of the covariance matrix, the random vector can be described in terms of a vector of independent random variables $\{\theta_i\}$, $i = 1, \ldots, 100$. Then, the vector of uncertain system parameters can be written as $\{\omega\}^t = \langle f_y, E, k_d, k_I, \theta_1, \ldots, \theta_{100}\rangle$, and the number of uncertain parameters involved in the problem is 104. The failure event F of interest is defined as failure of a series system of 300 secondary failure events. Then

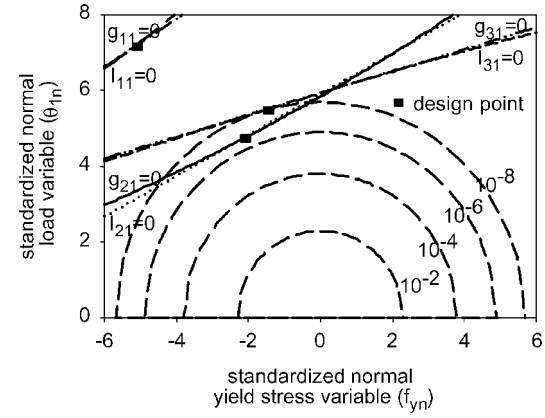

Fig. 2 Limit state surfaces and tangent hyperplanes at the design points projected into the load-yield stress space (initial design).

$$F = \bigcup_{j=1}^{3} \bigcup_{i=1}^{100} F_{ji} \tag{16}$$

where F_{ji} is a secondary failure event defined as $F_{ji} = \{\{\omega\} \in \Omega : g_{ji}(\{\omega\} \mid \{z\}) \leq 0\}$, $j = 1, 2, 3$, and $i = 1, \ldots, 100$. The specified level of failure for the system is taken equal to $P_F^* = 10^{-3}$.

Figure 2 shows some limit state surfaces corresponding to the initial design. A coefficient of correlation $\rho = 0.8$ for the elements of the axial load vector is used in this case. For illustration, the standard normal space corresponding to the uncertain system parameters is projected into the load-yield stress space (f_{yn}, θ_{1n}), where f_{yn} and θ_{1n} are the standardized normally distributed random variables corresponding to f_y and θ_1, respectively. The set of points (f_{yn}, θ_{1n}) where $g_{j1}(f_{yn}, E^*, k_d^*, k_I^*, \theta_{1n}, \theta_2^*, \ldots, \theta_{100}^*) = 0$, $j = 1, 2, 3$, and where $(\cdot)^*$ denotes the value of the uncertain parameter at the design point, is shown in Fig. 2. Some contours of the joint probability density function of f_{yn} and θ_{1n} are also shown in Fig. 2. It is seen that the failure surfaces $g_{j1} = 0$, $j = 1, 2, 3$, of the secondary failure events are well approximated by the tangent hyperplanes $l_{j1} = 0$, $j = 1, 2, 3$, around the design points. The radii of curvature of the failure surfaces at the design points are large compared to the distance from the surfaces to the origin. Similar results are obtained for the other limit state surfaces and for the cases when the elements of the load vector are fully correlated or completely uncorrelated. The first-order reliability method is able to capture the main contribution to the value of the failure probability of the secondary failure events. The same result is observed at the different designs obtained during the optimization process. These results also indicate that the failure regions corresponding to the secondary failure events F_{ji}, $j = 1, 2, 3$, and $i = 1, \ldots, 100$, are described in a simple way. In this case, the actual complexity of the test problem lies in the interaction of the 300 secondary failure events in forming the system failure event F.

Table 2 shows the final designs of the optimization process. The side constraints for the design variables are set as follows: $1.0 \leq d \leq 3.0$ m and $4.0 \times 10^{-3} \leq e \leq 15.0 \times 10^{-3}$ m. For these final designs, a high level of accuracy in the failure probability

Table 2 Final designs

ρ	d, m	e, m	Cost, C	No. of sublevel problems
1.0	1.29	9.49×10^{-3}	19.36×10^3	28
0.8	1.34	9.86×10^{-3}	20.83×10^3	27
0.0	1.36	10.03×10^{-3}	21.53×10^3	29
Initial design	1.50	11.00×10^{-3}	26.00×10^3	

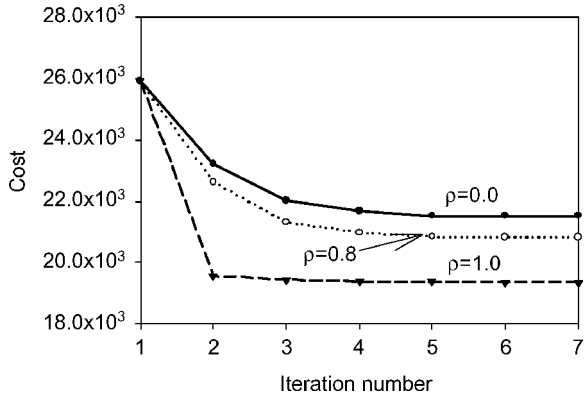

Fig. 3 Iteration histories.

estimates is used. The statistical error of the estimated failure probabilities is measured in terms of the COV, that is, the ratio of the standard deviation to the mean of the estimator $\mu_{\tilde{P}_F(\{z\})}$. This ratio is computed as

$$\frac{\sigma_{\tilde{P}_F(\{z\})}}{\mu_{\tilde{P}_F(\{z\})}} = \sqrt{\frac{1}{N(N-1)} \sum_{r=1}^{N} \left[\frac{\phi(\{\varpi\}_r) I_F(\{\varpi\}_r)}{f(\{\varpi\}_r)} - \tilde{P}_F(\{z\}) \right]^2} \Bigg/ \tilde{P}_F(\{z\}) \tag{17}$$

where the samples $\{\varpi\}_r, r = 1, \ldots, N$, are independent identically distributed samples simulated according to the importance sampling density $f(\cdot)$. A COV equal to 0.05 is used for the final designs shown in Table 2. The probability of failure of the initial design is equal to $P_F = 1.3 \times 10^{-9}$ for $\rho = 1.0$, $P_F = 6.8 \times 10^{-8}$ for $\rho = 0.8$, and $P_F = 1.4 \times 10^{-7}$ for $\rho = 0.0$. From Table 2, it is seen that the initial cost is decreased at the final design for all cases, showing the effectiveness of the design process in reducing the initial cost of the design. Table 2 also shows the effect of correlation of the axial loads on the final design. The results indicate that the effect of uncertainty in the axial loads is more significant in the completely uncorrelated case. That is, the effect of uncertainty in the axial loads tends to decrease as the elements of the load vector become correlated. This is reasonable because in the fully correlated case there are only three different secondary failure events that contribute to the system failure probability. In this case, only the three basic failure elements, that is, material yield, local buckling, and global buckling participate in the definition of the system failure event F because $F_{ji} = F_{jk}, \forall i, k, j = 1, 2, 3$. On the contrary, the number of secondary failure events contributing to the system failure probability in the uncorrelated case is equal to 300. These results suggest that the fully uncorrelated case may be taken as a conservative design if the correlation characteristics of the elements of the axial load are not well known.

The effectiveness of the optimization process corresponding to the three cases considered in Table 2 is shown in Fig. 3. In Fig. 3, the iteration history of the optimization process is presented. Observe that the optimization process converges in few iterations. In all cases, the convergence of the top-level optimization problem is obtained in fewer than seven design cycles. The total number of sublevel problems to be solved during the optimization process is shown in the last column of Table 2. Note that the entire design process requires fewer than 30 estimations of the system failure probability. On the other hand, the design points of the secondary failure events, which are needed in the sublevel problem, are the solution of minimization problems with 104 optimization variables [problem (5)]. Table 3 shows the number of evaluations of the limit state function involved in the calculation of the design point corresponding to the secondary failure event F_{11} at different design stages. A coefficient of correlation equal to $\rho = 0.8$ for the elements of the axial load vector is used in this case. A similar number of function evaluations is found for the calculation of the design points corresponding to the other secondary failure events. The cost of finding the design points during the whole optimization process represents approximately 60% of the total calculation cost of the design process. Note that each function evaluation is immediate in this case because the limit state functions are explicit in terms of the uncertain parameters. Therefore, it is expected that in more general cases the calculation of the design points will dominate the total calculation cost of the design process.

The accuracy and convergence of the importance sampling technique used during the optimization process are now explored. Figure 4 shows the importance sampling estimate as a function of the number of samples at the final design $\{\hat{z}\}$. The corresponding COV of the estimator is plotted in Fig. 5. Results using up to 800 samples are shown in Figs. 4 and 5. The dashed, dotted, and solid lines correspond to sampling histories for $\rho = 1.0$, 0.8, and 0.0, respectively. An exact solution for $P_F(\{\hat{z}\})$ is not feasible in this case due to the large dimension of the uncertain parameter space. Simulation results using 10^8 samples, however, show that they practically converge to the same value. For discussion purposes, such a value

Table 3 Number of function evaluations

Sequence of designs	No. of evaluations				
Initial			550		
Intermediates	420	400	316	301	205
Final			102		

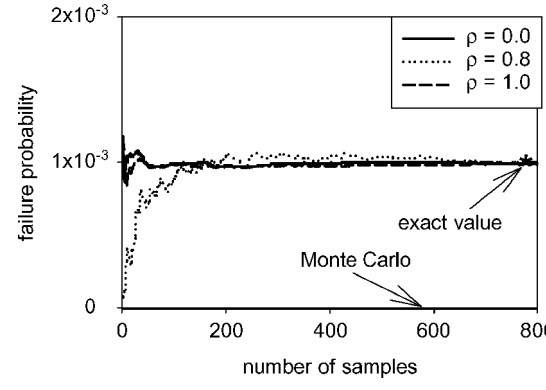

Fig. 4 Failure probability estimates at the final design.

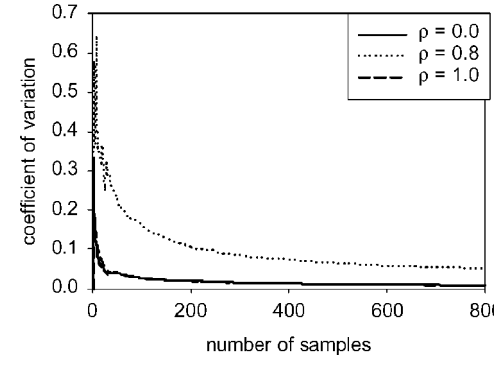

Fig. 5 COV of the failure probability estimates at the final design.

can be taken as the exact solution and is marked with an asterisk in Fig. 5. The failure probability estimate at the final design predicted by straightforward Monte Carlo simulation as a function of the number of samples is also shown in Fig. 4. The inefficiency of the basic Monte Carlo simulation procedure is evident from these results because there are no samples lying in the failure domain, and therefore, the probability of failure is zero. In this case, the first failure sample occurs after 10^4 simulations.

From Figs. 4 and 5, it is seen that the completely uncorrelated case converges very fast. A COV less than 0.1 is obtained in fewer than 10 samples. In this case, the three basic failure elements are independent at different times, and therefore, the correlation of the secondary failure events is controlled by the interaction among the three basic failure elements at each time. The overall interaction of the secondary failure events is found to be very small. Thus, the importance sampling quotient Q is close to unity, which leads to a small variation in the estimate $\tilde{P}_F(\{\hat{z}\})$. Similar behavior is observed in the fully correlated case, $\rho = 1.0$. In this situation, the number of secondary failure events is reduced to the three basic failure elements because the axial load is described by just one random variable. The small dimension of the reliability model in this case leads to a fast convergence to $P_F(\{\hat{z}\})$. The interaction of the secondary failure events is more intense in the intermediate case, that is, when the coefficient of correlation is equal to 0.8. In this case, the number of samples required to achieve a COV less than 0.1 is about 230, and only 30 samples are required to obtain a coefficient of variation less than 0.3. These results show that the number of samples required to achieve a small variability in the estimates is very small compared with those ordinarily obtained by general simulation techniques implying the efficiency of the sampling strategy. The same efficiency of the simulation procedure is observed during the entire design process. Therefore, the interaction of the secondary failure events is estimated in a very efficient manner.

The relatively small number of samples required in the simulation procedure, as well as the small number of sublevel problems to be solved during the design process, shows the efficiency and feasibility of the proposed formulation in the context of this example problem. Table 4 shows the final designs obtained when different levels of accuracy are used in the failure probability estimates. The level of accuracy in the probability estimates is measured by the COV. Four values for the COV are considered: 0.05, 0.1, 0.2, and 0.3. A coefficient of correlation equal to $\rho = 0.8$ for the elements of the axial load vector is used in Table 4. Note that the optimization process is relatively insensitive to the level of statistical error of the estimated failure probabilities. Therefore, a moderate level of accuracy in the failure probability estimate is enough to yield sufficiently accurate results for the final design, implying the robustness and efficiency of the design process.

Finally, to gain insight into the interaction of the secondary failure events, the relative importance of the secondary failure events at the initial and final designs is presented in Figs. 6 and 7, respectively. A coefficient of correlation $\rho = 0.8$ is considered in this case. The relative importance of the secondary failure event F_{ji} in contributing to the system failure probability is evaluated by the relative frequency of samples simulated from the conditional distribution $\phi(\{\varpi\}|F_{ji})$. Observe from Fig. 6 that the secondary failure events related to the local buckling failure modes are significantly more important than the other failure events at the initial design. The results also show that all of the secondary local buckling failure events are important in contributing to the failure probability. The interaction of the secondary failure events changes during the design process. In fact, it is seen from Fig. 7 that there is a strong interaction between the local

Table 4 Robustness of final design

COV, %	d, m	e, m
5	1.34	9.86×10^{-3}
10	1.34	9.86×10^{-3}
20	1.34	9.85×10^{-3}
30	1.34	9.84×10^{-3}

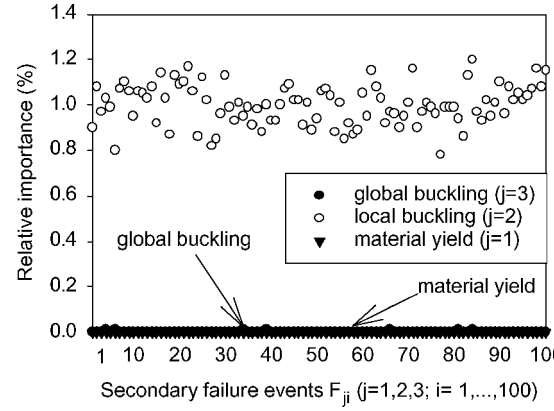

Fig. 6 Relative importance of the secondary failure events in contributing to the failure probability (initial design).

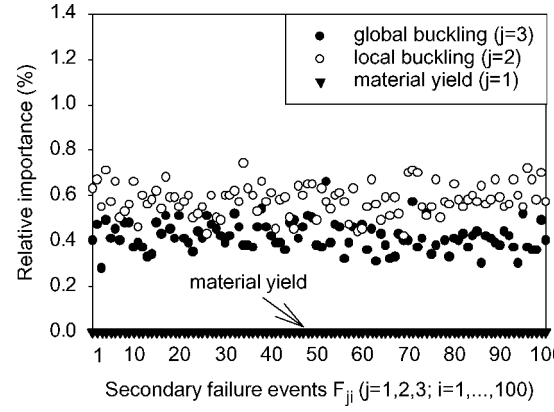

Fig. 7 Relative importance of the secondary failure events in contributing to the failure probability (final design).

and global buckling failure modes at the final design. In this case, the value of the failure probability is obtained almost exclusively from the contribution of the secondary failure events related to the stability failure modes. Therefore, the local and global buckling failure modes can be considered to be active at the final design. The results from Fig. 7 also show that all of the active secondary failure events at the final design are important in contributing to the estimate $\tilde{P}_F(\{\hat{z}\})$. From these results, it is clear that the stability failure modes govern the final design of the system. This information may be used, for example, to design a better resistance modeling of the steel column.

Conclusions

A design strategy with a class of reliability models of high dimension has been presented. A class of reliability problems modeled as a series system of a large number of secondary failure events is considered. It is assumed that each secondary event is sufficiently approximated by the half-space defined by the hyperplane that is tangent to the limit surface at the design point. The design process is written as a two-level nonlinear constrained optimization problem. The top level includes the overall optimization in the design variables, whereas the sublevel problem corresponds to the failure probability estimates. The formulation, which considers the uncertainty explicitly during the optimization process, results in robust designs that are optimal over the range of variable conditions under which most real systems must operate. The results of the test problem show the efficiency and feasibility of the proposed design process in the context of a reliability-based optimization problem of high dimension. The convergence of the top-level optimization problem is obtained in few design cycles. The combination of reliability evaluation methods based on first-order approximations and simulation allows an efficient solution of the sublevel problems during the

design process. The number of samples required for the estimation of the probability of failure is very small, even for the case of high level of accuracy in the estimates. As this particular yet illustrative example demonstrates, the proposed implementation provides a feasible tool for solving a complex reliability-based optimization problem. The proposed implementation is expected to be useful in the analysis, design, and optimization for the class of problems considered in this work.

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A. Messac
Associate Editor